SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3282. Correction. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain and Pantelimon George Popescu, Bucharest, Romania.*

Let $A(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a monic polynomial with complex coefficients. Suppose that $a_1 = -a_0$, and that the zeroes $z_1, z_2, \ldots, z_n$ of $A(z)$ are distinct, non-zero complex numbers. Prove that

$$\sum_{k=1}^{n} \frac{e^{z_k}}{z_k^2} \prod_{\substack{j=1\\ j\neq k}}^{n} \frac{1}{z_k - z_j} = 0 .$$

Counterexample by Oliver Geupel, Brühl, NRW, Germany.

The statement is false. As a counterexample consider the polynomial $A(z) = z^2 + \frac{1}{2}z - \frac{1}{2} = (z+1)\left(z - \frac{1}{2}\right)$, where $z_1 = -1$ and $z_2 = \frac{1}{2}$. We obtain

$$\sum_{k=1}^{n} \frac{e^{z_k}}{z_k^2} \prod_{\substack{j=1\\ j\neq k}}^{n} \frac{1}{z_k - z_j} = \frac{1}{e} \cdot \frac{1}{-1-\frac{1}{2}} + \frac{e^{\frac{1}{2}}}{\frac{1}{4}} \cdot \frac{1}{\frac{1}{2}+1}$$

$$= \frac{2\left(4e^{\frac{3}{2}} - 1\right)}{3e} \neq 0 .$$

Problem 3282 was originally misstated in [2007 : 429, 431]. *Michel Bataille, Rouen, France gave a counterexample to that earlier version, as did Geupel. The correction of 3282 appeared in* [2008 : 239, 241]. *Our apologies to all parties for not spotting the error initially.*

3263. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

The Fibonacci numbers F_n and Lucas numbers L_n are defined by the following recurrences:

$$F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_{n+1} = F_n + F_{n-1} \quad \text{for } n \geq 1;$$
$$L_0 = 2, \quad L_1 = 1, \quad \text{and} \quad L_{n+1} = L_n + L_{n-1} \quad \text{for } n \geq 1.$$

Prove that for each positive integer n,

$$L_n L_{n+1} \leq 2 + \left(\sum_{k=1}^{n} L_k F_{2k}\right)^{\frac{1}{2}} \cdot \sum_{k=1}^{n} \frac{L_k^2}{\sqrt{F_k}} .$$

I. *Solution by Oliver Geupel, Brühl, NRW, Germany.*

For each positive integer n we have $L_nL_{n+1} = 2 + \sum_{k=1}^{n} L_k^2$, as can be verified by induction. Since $L_k \geq 1$ and $F_{2k} \geq F_k$ for each k, the inequality thus follows:

$$\begin{aligned} L_nL_{n+1} &= 2 + \sum_{k=1}^{n} L_k^2 \leq 2 + \sum_{k=1}^{n} \left(\sqrt{L_kF_{2k}} \cdot \frac{L_k^2}{\sqrt{F_k}}\right) \\ &\leq 2 + \sum_{k=1}^{n} \left(\left(\sum_{j=1}^{n} L_jF_{2j}\right)^{\frac{1}{2}} \frac{L_k^2}{\sqrt{F_k}} \right) \\ &= 2 + \left(\sum_{k=1}^{n} L_kF_{2k}\right)^{\frac{1}{2}} \cdot \sum_{k=1}^{n} \frac{L_k^2}{\sqrt{F_k}} . \end{aligned}$$

Equality holds if and only if $n = 1$.

II. *Solution by Arkady Alt, San Jose, CA, USA.*

The inequality can be strengthened. For each $k \geq 1$ let $a_k = L_k^{\frac{2}{3}} F_k^{\frac{1}{3}}$ and let $b_k = L_k^{\frac{4}{3}} F_k^{\frac{-1}{3}}$. Applying Hölder's Inequality

$$\sum_{k=1}^{n} a_kb_k \leq \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} b_k^q\right)^{\frac{1}{q}}$$

for $p = 3$ and $q = \frac{3}{2}$, we successively obtain

$$\begin{aligned} \sum_{k=1}^{n} L_k^2 &\leq \left(\sum_{k=1}^{n} L_k^2F_k\right)^{\frac{1}{3}} \left(\sum_{k=1}^{n} \frac{L_k^2}{\sqrt{F_k}}\right)^{\frac{2}{3}} , \\ \left(\sum_{k=1}^{n} L_k^2\right)^{\frac{3}{2}} &\leq \left(\sum_{k=1}^{n} L_k^2F_k\right)^{\frac{1}{2}} \cdot \left(\sum_{k=1}^{n} \frac{L_k^2}{\sqrt{F_k}}\right) . \end{aligned}$$

Since

$$\begin{aligned} \sum_{k=1}^{n} L_k^2 &= \sum_{k=1}^{n} L_k(L_{k+1} - L_{k-1}) \\ &= \sum_{k=1}^{n} (L_kL_{k+1} - L_kL_{k-1}) = L_nL_{n+1} - 2 \end{aligned}$$

and it is well known that $L_kF_{2k} = L_k^2F_k$ for each k, we then have

$$(L_nL_{n+1} - 2)^{\frac{3}{2}} \leq \left(\sum_{k=1}^{n} L_kF_{2k}\right)^{\frac{1}{2}} \cdot \sum_{k=1}^{n} \frac{L_k^2}{\sqrt{F_k}} .$$